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Material instabilities in inelastic saturated porous media under dynamic loadings

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Abstract

Instabilities in inelastic saturated porous media are investigated here for general three-dimensional states under dynamic loadings using a perturbation approach.

Under quasi-static conditions, unbounded growth of perturbations is related to the emergence of stationary discontinuities under drained or undrained conditions, while under dynamic conditions, unbounded growth is related either to the emergence of stationary discontinuities (and these are set by drained conditions) or to the appearance of the flutter phenomenon (acceleration waves).

For associative behaviour unbounded growth always corresponds to localization under drained conditions and the onset of growth of perturbations occurs here only through divergence growth. It is only for non-associative flow that unbounded growth may correspond to undrained localization in quasi-static conditions and to flutter under dynamic conditions. © 2002 Elsevier Science Ltd. All rights reserved.

1. Introduction

The linear perturbation approach has been used to analyze instability phenomena in several contexts. In particular, some applications of this methodology to inelastic saturated porous media, mainly in geomechanics, are available in the literature. Thus Rice (1975) analyzed a fluid saturated layer under quasi-static combined shear and compression. For the dilatant behaviour that he considered, he showed that the homogeneous undrained response of the layer first becomes unstable when the condition of localization of deformation was satisfied for the drained deformation. Instability there was understood as the exponential growth of spatial non-uniformities. Rice's analysis was later extended to other situations in Rudnicki (1983), Vardoulakis (1996a,b), Rudnicki (2000) and to include inertia in Vardoulakis (1986). With a different approach, namely by analyzing acceleration waves, Loret (1990) and Loret and Harireche (1991) showed that stationary discontinuities (acceleration waves with vanishing speed) first become available

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when the condition of localization is met for the drained deformation, irrespective of the state of stress. This result was shown to hold also for anisotropic saturated porous media with double porosity in Rizzi and Loret (1999). Further, Loret et al. (1997) analyzed the growth and decay of acceleration waves. The results obtained there were continued in Simoes et al. (1999) by studying dynamic instabilities in elastic porous media comparing the acceleration wave analysis and the harmonic waves analysis.

In a recent paper (Benallal and Comi, submitted for publication), the authors have analyzed the development and growth of instabilities in inelastic saturated porous media subjected to quasi-static loadings. More precisely, using a perturbation approach, they obtained for a large class of constitutive equations and for three-dimensional states of stress the conditions for the onset of growth and also the conditions for unbounded growth of perturbations. The analysis is continued here to include inertia and to extend the results to dynamic loadings. The comprehension of the development of instabilities is indeed very important in many practical situations such as transient phenomena in earthquakes.

The body of the paper is structured as follows: in Section 2, the constitutive and evolution equations for elastic–plastic saturated media following Biot (1955) and Coussy (1995) will be described. This is done in the framework of continuum thermodynamics with internal variables. In Section 3 we briefly recall for the sake of comparison acceleration and harmonic wave analyses. We give alternative forms to the characteristic equations for wave speeds as given in Simoes et al. (1999) thus allowing for an easier interpretation and we bring some complements to their analysis.

Section 4 contains the application of the perturbation approach to elastic–plastic saturated porous media. A homogeneously deformed body (assumed infinite here for simplicity) with uniform properties is considered. At a given stage of its deformation path, an infinitely small perturbation is applied to the solid and the perturbed deformation is analyzed.

In Section 5 we study conditions for unbounded growth of perturbations. This unbounded growth is shown to be related to the emergence of acceleration waves with non-real wave speeds. The onset of unbounded growth corresponds therefore either to the appearance of stationary discontinuities or to the flutter phenomenon.

In Section 6 the onset of growth of perturbations is considered. For a quite general type of non-linear plastic potentials (including most of the models available in the literature) the growth condition is derived in closed forms. Conditions for transition from decaying behaviour to growing behaviour of perturbations are obtained. Both divergence growth and transition through flutter-type instabilities are analyzed.

Finally, Section 7 is devoted to the application of the results to a specific set of constitutive equations; namely saturated porous media with skeleton obeying Drucker–Prager like constitutive behaviour are considered.

2. Dynamic evolution of saturated porous media

2.1. Field equations

Consider the dynamic evolution of a saturated porous body of volume V and boundary S consisting of a porous solid skeleton (s) and of the filled fluid (f). The governing equations of the coupled solid–fluid system are given by the field equations of dynamic equilibrium, conservation of mass and compatibility supplemented by constitutive relations.

Let us introduce the following quantities: ϕ is the porosity; ρ_s, ρ_f are microscopic densities of the solid phase and of the fluid phase; $\rho = (1 - \phi)\rho_s + \phi\rho_f$ is density of the assembly; $\boldsymbol{u}_s$ is skeleton displacement field; $\boldsymbol{w}$ is pseudo-displacement of the fluid relative to the skeleton, such that $\dot{\boldsymbol{w}}$ is the fluid flux ($\dot{w}_i$ being the fluid volume crossing in the time unit the unit surface normal to the ith axis); $\boldsymbol{u}_f = \boldsymbol{u}_s + (1/\phi)\boldsymbol{w}$ is total displacement of the fluid; $\boldsymbol{\epsilon}$ is strain in the skeleton; ζ is variation of fluid content (i.e. the volume change of fluid per unit volume of mixture also defined by the ratio m/ρ_f, m being the mass variation); $\boldsymbol{\sigma}$ is total Cauchy stress in the combined solid and fluid mix and p is pore fluid pressure.

Assuming small strains in the solid, the geometric compatibility conditions are:

$$\boldsymbol{\epsilon} = \frac{1}{2}\left(\operatorname{grad}\boldsymbol{u}_s + \operatorname{grad}^T\boldsymbol{u}_s\right) \tag{1}$$

The equation of motion of the total system is:

$$\operatorname{div}\boldsymbol{\sigma} + \rho\boldsymbol{b} = \rho\ddot{\boldsymbol{u}}_s + \rho_f\ddot{\boldsymbol{w}} \tag{2}$$

$\boldsymbol{b}$ being the body force acceleration so that $\rho\boldsymbol{b}$ is the body force of the mixture solid–fluid.

The equation of motion of the fluid specifies the relative movement of the fluid with respect to the skeleton. Assuming the classical linear Darcy's law for the viscous drag, this equation reads:

$$-\operatorname{grad}p + \rho_f\boldsymbol{b} = \rho_f\left(\ddot{\boldsymbol{u}}_s + \frac{\ddot{\boldsymbol{w}}}{\phi}\right) + \frac{\dot{\boldsymbol{w}}}{k} \tag{3}$$

where k is the permeability under isotropic conditions, assumed constant in this work.

The final equation is supplied by the conservation of mass and reads:

$$\operatorname{div}\dot{\boldsymbol{w}} + \dot{\zeta} = 0 \tag{4}$$

Combination of (3) and (4) yields:

$$\dot{\zeta} - k\left[\nabla^2 p - \operatorname{div}\rho_f\boldsymbol{b} + \operatorname{div}\rho_f\left(\ddot{\boldsymbol{u}}_s + \frac{\ddot{\boldsymbol{w}}}{\phi}\right)\right] = 0 \tag{5}$$

Using the relation between relative and total displacement of the fluid:

$$\boldsymbol{w} = \phi(\boldsymbol{u}_f - \boldsymbol{u}_s) \tag{6}$$

Eqs. (2), (3) and (5) can be rewritten in terms of the total displacement of the fluid $\boldsymbol{u}_f$ instead of the relative displacement $\boldsymbol{w}$ as:

$$\operatorname{div}\boldsymbol{\sigma} + \rho\boldsymbol{b} - (1-\phi)\rho_s\ddot{\boldsymbol{u}}_s - \phi\rho_f\ddot{\boldsymbol{u}}_f = \boldsymbol{0} \tag{7}$$

$$\operatorname{grad}p - \rho_f\boldsymbol{b} + \rho_f\ddot{\boldsymbol{u}}_f + \frac{\phi}{k}(\dot{\boldsymbol{u}}_f - \dot{\boldsymbol{u}}_s) = \boldsymbol{0} \tag{8}$$

$$\dot{\zeta} - k[\nabla^2 p - \operatorname{div}\rho_f(\boldsymbol{b} - \ddot{\boldsymbol{u}}_f)] = 0 \tag{9}$$

The initial boundary value problem is completed by initial and boundary conditions.

Remark 2.1. The inertial term of the fluid $\rho_f\ddot{\boldsymbol{u}}_f$ in Eq. (7) can be substituted by its value computed from Eq. (8), thus obtaining

$$\operatorname{div}\boldsymbol{\sigma} + \phi\operatorname{grad}p + (1-\phi)\rho_s\boldsymbol{b} - (1-\phi)\rho_s\ddot{\boldsymbol{u}}_s + \frac{\phi^2}{k}(\dot{\boldsymbol{u}}_f - \dot{\boldsymbol{u}}_s) = 0 \tag{10}$$

Eqs. (10) and (8) can be interpreted directly as the equations of motion of the solid and of the fluid phase, respectively. They have been here derived as proposed in Zienkiewicz and Shiomi (1984), but can also be directly derived from the theory of mixtures (Truesdell and Toupin, 1960, Bowen, 1976).

Remark 2.2. For slow dynamic processes as those occurring in seismic analyses the inertial terms of the fluid can be neglected. The corresponding governing equations can be obtained e.g. from Eqs. (10) and (8) by setting $\rho_f = 0$. The quasi-static case is recovered for $\rho_s = \rho_f = 0$. In these cases, a formulation in which the only independent variables are $\boldsymbol{u}_s$ and p is preferred: the viscous drag term is computed from Eq. (8) and substituted into Eq. (10).

2.2. Constitutive relations

Following the Biot formulation (Biot, 1955), the constitutive law is given in this section in terms of relations between the static variables (total Cauchy stress $\boldsymbol{\sigma}$ and the pore fluid pressure p) and the conjugate kinematic variables (strain of the skeleton $\boldsymbol{\epsilon}$ and variation of fluid content ζ). In the poro-plastic case, see e.g. Coussy (1995), the kinematic variables are partitioned into elastic and plastic parts:

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^e + \boldsymbol{\epsilon}^p \tag{11}$$

$$\zeta = \zeta^e + \zeta^p \tag{12}$$

and two additional conjugate variables $\boldsymbol{\chi}$ and $\boldsymbol{\alpha}$ are introduced to describe various dissipative phenomena, $\boldsymbol{\alpha}$ being a vector of internal variables and $\boldsymbol{\chi}$ the associated thermodynamic force.

The stress $\boldsymbol{\sigma}$, the pore pressure p and the thermodynamical forces $\boldsymbol{\chi}$ are given by the state equations:

$$\boldsymbol{\sigma} = \boldsymbol{E}^d : \boldsymbol{\epsilon}^e + bM[b\,\mathrm{Tr}\,(\boldsymbol{\epsilon}^e) - \zeta^e]\mathbf{1} \tag{13}$$

$$p = M[\zeta^e - b\,\mathrm{Tr}\,(\boldsymbol{\epsilon}^e)] \tag{14}$$

$$\boldsymbol{\chi} = -\boldsymbol{h} \cdot \boldsymbol{\alpha} \tag{15}$$

where $\boldsymbol{E}^d$ is the drained elastic tensor, M is the Biot modulus, b is the Biot coefficient of effective stress, $\mathrm{Tr}\,(\cdot)$ is the trace operator.

Using (14), Eq. (13) can be written in the classical form used by Biot:

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} + bp\mathbf{1} = \boldsymbol{E}^d : \boldsymbol{\epsilon}^e \tag{16}$$

in which $\boldsymbol{\sigma}'$ is the Biot effective stress which generalizes the Terzaghi effective stress and reduces to it when $b = 1$.

In the following we will assume isotropic elastic behaviour, i.e.:

$$\boldsymbol{E}^d = \left(K^d - \frac{2G}{3}\right)\mathbf{1} \otimes \mathbf{1} + 2G\boldsymbol{I} \tag{17}$$

K^d and G are respectively the bulk and shear moduli of the drained material. We also introduce for later use v the drained Poisson's ratio. In (17), $\mathbf{1}$ and $\boldsymbol{I}$ are respectively the second-order and fourth-order symmetric identity tensors.

The elastic response of the porous medium is then defined by four independent parameters. In the above relations we have chosen the two drained moduli K^d and G and the Biot's parameters b and M. Other elastic constants, of easier experimental identification, can be introduced (see e.g. Nur and Byerlee, 1971, Rice and Cleary, 1976); for instance a noteworthy alternative to the Biot's parameters b and M is represented by the Skempton coefficient B and the undrained bulk modulus K^u. Moreover, the macroscopic elastic constant above defined can also be expressed in terms of the micromechanical parameters such as stiffness of the solid grains, porosity, etc, see e.g. Cheng and Detournay (1993). The evolution of the internal variables is given by introducing the plastic potential $F(\boldsymbol{\sigma}, p, \boldsymbol{\chi})$ and read:

$$\dot{\boldsymbol{\epsilon}}^{\mathrm{p}} = \dot{\lambda}\frac{\partial F}{\partial \boldsymbol{\sigma}}, \quad \dot{\zeta}^{\mathrm{p}} = \dot{\lambda}\frac{\partial F}{\partial p}, \quad \dot{\boldsymbol{\alpha}} = \dot{\lambda}\frac{\partial F}{\partial \boldsymbol{\chi}} \tag{18}$$

where $\dot{\lambda}$ is the plastic multiplier satisfying the classical Kuhn–Tucker relations:

$$\dot{\lambda} \geqslant 0, \ f \leqslant 0, \ \dot{\lambda} f = 0, \tag{19}$$

$f = f(\boldsymbol{\sigma}, p, \boldsymbol{\chi})$ being the yield function. The dependence of f and F on $\boldsymbol{\sigma}$ and p is left arbitrary here although it is usually done through an effective stress. For instance the use of Terzaghi's effective stress $\boldsymbol{\sigma} + p\mathbf{1}$ is proposed and justified in some instances (Rice, 1977).

Assuming that the material is in plastic loading everywhere, using relations (13)–(16), (18) and the consistency condition ($\dot{f} = 0$), the rate constitutive equations may be given the following two alternative forms:

$$\begin{cases} \dot{\boldsymbol{\sigma}} = \boldsymbol{H}^{\mathrm{d}} : \dot{\boldsymbol{\epsilon}} - \boldsymbol{K}\dot{p} \\ \dot{\zeta} = \boldsymbol{L} : \dot{\boldsymbol{\epsilon}} + D\dot{p} \end{cases} \tag{20}$$

$$\begin{cases} \dot{\boldsymbol{\sigma}} = \boldsymbol{H}^{\mathrm{u}} : \dot{\boldsymbol{\epsilon}} - \frac{1}{D}\boldsymbol{K}\dot{\zeta} \\ \dot{p} = \frac{1}{D}\boldsymbol{L} : \dot{\boldsymbol{\epsilon}} + \frac{1}{D}\dot{\zeta} \end{cases} \tag{21}$$

In Eq. (20) $\boldsymbol{H}^{\mathrm{d}}$ is the drained tangent modulus relating the strain rate to the stress rate under drained conditions ($\dot{p} = 0$)

$$\boldsymbol{H}^{\mathrm{d}} = \boldsymbol{E}^{\mathrm{d}} - \frac{\boldsymbol{E}^{\mathrm{d}} : \frac{\partial F}{\partial \boldsymbol{\sigma}} \otimes \frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{E}^{\mathrm{d}}}{H^{\mathrm{d}}} \tag{22}$$

with

$$H^{\mathrm{d}} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{E}^{\mathrm{d}} : \frac{\partial F}{\partial \boldsymbol{\sigma}} + \frac{\partial f}{\partial \boldsymbol{\chi}} \cdot \boldsymbol{h} \cdot \frac{\partial F}{\partial \boldsymbol{\chi}} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{E}^{\mathrm{d}} : \frac{\partial F}{\partial \boldsymbol{\sigma}} + h^{\mathrm{d}} \tag{23}$$

In Eq. (21) $\boldsymbol{H}^{\mathrm{u}}$ is the undrained tangent modulus relating the strain rate to the stress rate under undrained conditions ($\dot{\zeta} = 0$) and given by

$$\boldsymbol{H}^{\mathrm{u}} = \boldsymbol{H}^{\mathrm{d}} + \frac{\boldsymbol{K} \otimes \boldsymbol{L}}{D} \tag{24}$$

$\boldsymbol{K}$ and $\boldsymbol{L}$ are second order tensors and D is a scalar defined respectively by

$$\boldsymbol{K} = b\mathbf{1} + \left(\frac{\frac{\partial f}{\partial p} - b\frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbf{1}}{H^{\mathrm{d}}} \right) \boldsymbol{E}^{\mathrm{d}} : \frac{\partial F}{\partial \boldsymbol{\sigma}} \tag{25}$$

$$\boldsymbol{L} = b\mathbf{1} + \left(\frac{\frac{\partial F}{\partial p} - b\frac{\partial F}{\partial \boldsymbol{\sigma}} : \mathbf{1}}{H^{\mathrm{d}}} \right) \frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{E}^{\mathrm{d}} \tag{26}$$

$$D = \frac{1}{M} + \frac{\left(\frac{\partial F}{\partial p} - b\frac{\partial F}{\partial \boldsymbol{\sigma}} : \mathbf{1}\right)\left(\frac{\partial f}{\partial p} - b\frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbf{1}\right)}{H^{\mathrm{d}}} \tag{27}$$

As shown in Benallal and Comi (submitted for publication) D is always positive if snap-back is excluded.

Some algebraic manipulations allow to write (24) in the more convenient form (similar to (22))

$$\boldsymbol{H}^{\mathrm{u}} = \boldsymbol{E}^{\mathrm{u}} - \frac{\boldsymbol{E}^{\mathrm{u}} : \left(\frac{\partial F}{\partial \boldsymbol{\sigma}} - \frac{bM}{3K^{\mathrm{u}}}\frac{\partial F}{\partial p}\mathbf{1}\right) \otimes \left(\frac{\partial f}{\partial \boldsymbol{\sigma}} - \frac{bM}{3K^{\mathrm{u}}}\frac{\partial f}{\partial p}\mathbf{1}\right) : \boldsymbol{E}^{\mathrm{u}}}{H^{\mathrm{u}}} \tag{28}$$

where $\boldsymbol{E}^{\mathrm{u}}$ is the *undrained* elastic tensor:

$$\boldsymbol{E}^{\mathrm{u}} = \boldsymbol{E}^{\mathrm{d}} + b^2 M \mathbf{1} \otimes \mathbf{1} = \left(K^{\mathrm{u}} - \frac{2G}{3}\right) \mathbf{1} \otimes \mathbf{1} + 2G\boldsymbol{I} \tag{29}$$

and

$$K^{\mathrm{u}} = K^{\mathrm{d}} + b^2 M \tag{30}$$

$$H^{\mathrm{u}} = h^{\mathrm{u}} + \left(\frac{\partial f}{\partial \boldsymbol{\sigma}} - \frac{bM}{3K^{\mathrm{u}}}\frac{\partial f}{\partial p}\mathbf{1}\right) : \boldsymbol{E}^{\mathrm{u}} : \left(\frac{\partial F}{\partial \boldsymbol{\sigma}} - \frac{bM}{3K^{\mathrm{u}}}\frac{\partial F}{\partial p}\mathbf{1}\right) \tag{31}$$

$$h^{\mathrm{u}} = h^{\mathrm{d}} + M\frac{K^{\mathrm{d}}}{K^{\mathrm{u}}}\frac{\partial f}{\partial p}\frac{\partial F}{\partial p} \tag{32}$$

h^{d} and h^{u} are respectively the drained and undrained plastic hardening moduli, K^{u} the undrained bulk modulus.

Remark 2.3. In some instances (see e.g. Zhang et al. (1999)) the plastic potential is assumed to be a function of stresses and pore pressure only through the same effective stress $\boldsymbol{\sigma}'$ governing the elastic behaviour: $F = F(\boldsymbol{\sigma}', \boldsymbol{\chi})$. In this case:

$$\dot{\zeta}^{\mathrm{p}} = \dot{\lambda} b \frac{\partial F}{\partial \boldsymbol{\sigma}'} : \mathbf{1} = b\,\mathrm{Tr}\,(\dot{\boldsymbol{\epsilon}}^{\mathrm{p}}) \tag{33}$$

and from Eqs. (14), (11) and (12) the pressure increment can be expressed in terms of total (elastic plus plastic) kinematic quantities as:

$$\dot{p} = M(\dot{\zeta} - b\,\mathrm{Tr}\,(\dot{\boldsymbol{\epsilon}})). \tag{34}$$

3. Acceleration waves, harmonic waves

Accelerations waves and harmonic waves in saturated porous media were investigated by Loret and his coworkers in a series of papers. Thus accelerations waves and the possibility of flutter instabilities were given in Loret and Harireche (1991) where they showed in particular that stationary discontinuities (acceleration waves with vanishing speed) first become available when the condition of localization is met for the drained deformation, irrespective of the state of stress. This result was shown to hold also for anisotropic saturated porous media with double porosity in Rizzi and Loret (1999). Further, Loret et al. (1997) analyzed the growth and decay of acceleration waves. The results obtained there were continued in Simoes et al. (1999) by studying dynamic instabilities in elastic porous media comparing the acceleration wave analysis and the harmonic waves analysis.

3.1. Acceleration waves

Acceleration waves are singular surfaces of order two across which both acceleration and velocity gradient are discontinuous. Stationary discontinuities correspond to the particular case where only the velocity gradient suffers a jump (Hill, 1962). We denote by $[[\cdot]]$ the jump across the wavefront of any quantity and by $\boldsymbol{n}$ the normal to this front. Across an acceleration wave, through Hadamard compatibility conditions, the jumps of velocity gradient and acceleration of the solid and the fluid satisfy respectively

$$\left[\left[\frac{\partial \boldsymbol{v}_s}{\partial \boldsymbol{x}}\right]\right] = \boldsymbol{g}_s \otimes \boldsymbol{n}, \quad \left[\left[\frac{\partial \boldsymbol{v}_s}{\partial t}\right]\right] = -c\boldsymbol{g}_s \tag{35}$$

$$\left[\left[\frac{\partial \boldsymbol{v}_f}{\partial \boldsymbol{x}}\right]\right] = \boldsymbol{g}_f \otimes \boldsymbol{n}, \quad \left[\left[\frac{\partial \boldsymbol{v}_f}{\partial t}\right]\right] = -c\boldsymbol{g}_f \tag{36}$$

$\boldsymbol{v}_s$ and $\boldsymbol{v}_f$ are the velocities in the solid and in the fluid respectively, $\boldsymbol{g}_s$ and $\boldsymbol{g}_f$ are the normal jumps of the velocity gradient and c is the speed of propagation of the wave.

We assume that body forces are continuous across the wavefront.

Using balance of momentum of the mixture (7), the rate constitutive Eq. (20) and Hadamard compatibility conditions, one has

$$[\boldsymbol{n} \cdot \boldsymbol{H}^d \cdot \boldsymbol{n}] \cdot \boldsymbol{g}_s - (\boldsymbol{K} \cdot \boldsymbol{n})[[\dot{p}]] - (1-\phi)\rho_s c^2 \boldsymbol{g}_s - \phi \rho_f c^2 \boldsymbol{g}_f = \boldsymbol{0} \tag{37}$$

Balance of momentum of the fluid (8) gives then

$$[[\mathrm{grad}\, p]] = \rho_f c \boldsymbol{g}_f \tag{38}$$

from which the jump of the pressure rate is obtained (again using Hadamard compatibility) as

$$[[\dot{p}]] = -\rho_f c^2 (\boldsymbol{g}_f \cdot \boldsymbol{n}) \tag{39}$$

Finally balance of mass (4) combined to the rate equations (21) and (6) gives

$$[[\dot{\zeta}]] = \boldsymbol{L} : \left[\left[\frac{\partial \boldsymbol{v}_s}{\partial \boldsymbol{x}}\right]\right] + D[[\dot{p}]] = -\phi(\boldsymbol{g}_f \cdot \boldsymbol{n} - \boldsymbol{g}_s \cdot \boldsymbol{n}) \tag{40}$$

Combining (39) and (40), one can compute the normal fluid displacement jump $\boldsymbol{g}_f \cdot \boldsymbol{n}$ only in terms of the solid displacement jump $\boldsymbol{g}_s$. Reporting the result in (39), one has also the jump of the pressure rate only in terms of $\boldsymbol{g}_s$

$$[[\dot{p}]] = -\frac{(\boldsymbol{L} - \phi \boldsymbol{1}) \cdot \boldsymbol{n}}{D - \frac{\phi}{\rho_f c^2}} \boldsymbol{g}_s \tag{41}$$

Finally putting these last two results in (37), and since normal jumps have been considered, one gets the following relation for the normal jump of the solid velocity gradient $\boldsymbol{g}_s$:

$$\left[\boldsymbol{n} \cdot \boldsymbol{H}^d \cdot \boldsymbol{n} + \frac{(\boldsymbol{K} - \phi \boldsymbol{1}) \cdot \boldsymbol{n} \otimes (\boldsymbol{L} - \phi \boldsymbol{1}) \cdot \boldsymbol{n}}{D - \frac{\phi}{\rho_f c^2}} - R_s c^2 \boldsymbol{1}\right] \cdot \boldsymbol{g}_s = \boldsymbol{0} \tag{42}$$

This gives the characteristic equation for the wave speeds of acceleration waves, i.e.

$$\det \left[\boldsymbol{n} \cdot \boldsymbol{H}^d \cdot \boldsymbol{n} + \frac{(\boldsymbol{K} - \phi \boldsymbol{1}) \cdot \boldsymbol{n} \otimes (\boldsymbol{L} - \phi \boldsymbol{1}) \cdot \boldsymbol{n}}{D - \frac{\phi}{\rho_f c^2}} - R_s c^2 \boldsymbol{1}\right] = 0 \tag{43}$$

where we have defined the macroscopic mass density of the solid

$$R_s = (1-\phi)\rho_s \tag{44}$$

As in Loret (1990), Loret and Harireche (1991), putting $c = 0$ in (43), one gets the condition for stationary discontinuities

$$\det[\boldsymbol{n} \cdot \boldsymbol{H}^d \cdot \boldsymbol{n}] = 0 \tag{45}$$

These stationary discontinuities are therefore available for the saturated porous material when localization would occur in the underlying drained solid.

3.2. Harmonic waves

Here we seek solutions of the initial boundary value problem in form the of plane waves, i.e

$$\begin{bmatrix} \boldsymbol{v}_{\mathrm{s}} \\ \boldsymbol{v}_{\mathrm{f}} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{v}}_{\mathrm{s}} \\ \bar{\boldsymbol{v}}_{\mathrm{f}} \\ \bar{p} \end{bmatrix} \exp\left[\mathrm{i}\xi(\boldsymbol{n}\cdot\boldsymbol{x} - ct)\right] \tag{46}$$

where $\boldsymbol{n}$ is the direction of propagation, ξ is the wave number and c is the wave speed. Substituting (46) in the rate form of the balance of momentum Eqs. (7), (8) and in the balance of mass (4) using the constitutive rate equation (20), one obtains

$$\begin{bmatrix} -\xi^2\left(\boldsymbol{n}\cdot\boldsymbol{H}^{\mathrm{d}}\cdot\boldsymbol{n} - R_{\mathrm{s}}c^2\mathbf{1}\right) & \phi\rho_{\mathrm{f}}\xi^2c^2\mathbf{1} & -\mathrm{i}\xi(\boldsymbol{K}\cdot\boldsymbol{n}) \\ \frac{\mathrm{i}\xi c\phi}{k}\mathbf{1} & -\left(\rho_{\mathrm{f}}\xi^2c^2 + \frac{\mathrm{i}\xi c\phi}{k}\right)\mathbf{1} & \mathrm{i}\xi\boldsymbol{n} \\ \mathrm{i}\xi(\boldsymbol{L}\cdot\boldsymbol{n} - \phi\boldsymbol{n}) & \mathrm{i}\xi\phi\boldsymbol{n} & D \end{bmatrix} \begin{Bmatrix} \bar{\boldsymbol{v}}_{\mathrm{s}} \\ \bar{\boldsymbol{v}}_{\mathrm{f}} \\ \bar{p} \end{Bmatrix} = \mathbf{0} \tag{47}$$

Eliminating the pressure and fluid displacement amplitudes from the second and last equations in (47) and upon substitution of these in the first Eq. (47), one gets the following relation for the solid velocity amplitude $\bar{\boldsymbol{v}}_{\mathrm{s}}$:

$$\left[\boldsymbol{n}\cdot\boldsymbol{H}^{\mathrm{d}}\cdot\boldsymbol{n} + \frac{(\boldsymbol{K} - \phi\overline{\Omega}\mathbf{1})\cdot\boldsymbol{n}\otimes(\boldsymbol{L} - \phi\overline{\Omega}\mathbf{1})\cdot\boldsymbol{n}}{D - \frac{\phi}{\rho_{\mathrm{f}}c^2}\overline{\Omega}} - \overline{R}c^2\mathbf{1}\right]\cdot\bar{\boldsymbol{v}}_{\mathrm{s}} = 0 \tag{48}$$

where the non-dimensional parameter $\overline{\Omega}$ and the "density like" parameter $\overline{R}$, both depending on the wave number ξ, have been defined

$$\overline{\Omega} = \frac{\mathrm{i}\xi c\rho_{\mathrm{f}}k}{\mathrm{i}\xi c\rho_{\mathrm{f}}k - \phi} \tag{49}$$

$$\overline{R} = R_{\mathrm{s}} + \frac{\phi^2\rho_{\mathrm{f}}}{\phi - \mathrm{i}\xi c\rho_{\mathrm{f}}k} \tag{50}$$

Note that the parameter $\overline{R}$, for ξ tending to infinity, tends to the macroscopic mass density of the solid R_{s}, while for ξ tending to zero it tends to the density of the mixture ρ.

From Eq. (48) one obtains the characteristic equation for the wave speeds of harmonic waves, i.e.

$$\det\left[\boldsymbol{n}\cdot\boldsymbol{H}^{\mathrm{d}}\cdot\boldsymbol{n} + \frac{(\boldsymbol{K} - \phi\overline{\Omega}\mathbf{1})\cdot\boldsymbol{n}\otimes(\boldsymbol{L} - \phi\overline{\Omega}\mathbf{1})\cdot\boldsymbol{n}}{D - \frac{\phi}{\rho_{\mathrm{f}}c^2}\overline{\Omega}} - \overline{R}c^2\mathbf{1}\right] = 0 \tag{51}$$

One immediately sees that when either the permeability k or the wave number ξ goes to infinity, $\overline{\Omega}$ tends to one, $\overline{R}$ tends to R_{s} and the characteristic equation for harmonic waves (51) reduces exactly to that of acceleration waves (43), as already pointed out in Simoes et al. (1999).

On the other hand, when the wave number $\xi \to 0$ (long wavelength limit) or the permeability $k \to 0$, $\overline{\Omega} \to 0$, $\overline{R} \to \rho$ and, (when $c \neq 0$) the following condition is obtained for the harmonic waves

$$\det\left[\boldsymbol{n}\cdot\boldsymbol{H}^{\mathrm{u}}\cdot\boldsymbol{n} - \rho c^2\mathbf{1}\right] = 0 \tag{52}$$

and this corresponds to the characteristic equation for acceleration waves for the undrained solid.

4. Perturbation analysis

We consider an infinite poro-elastic–plastic medium and assume uniform physical properties within it. This body is remotely and uniformly loaded in such a way that a homogeneous solution in terms of stresses and strains prevails throughout it.

In this section to detect instabilities a perturbation approach is used. An infinitesimal perturbation is superposed to the solution at a generic instant of the evolution and the behaviour of the perturbation is analyzed. Stability is assured if small perturbations produce only limited changes in the solution.

We denote by a superscript 0 all the fields corresponding to the homogeneous solution. This solution is such that $\nabla p^0 = \mathbf{0}$. To investigate its stability we superpose to it at a generic instant an infinitesimal perturbation denoted by δ and we analyze the behaviour of the perturbed fields $\boldsymbol{u} = \boldsymbol{u}^0 + \delta\boldsymbol{u}$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^0 + \delta\boldsymbol{\sigma}$, etc.

4.1. Non-linear initial boundary-value problem for the perturbation field

Using the fact that the reference solution satisfies all field and constitutive equations, the perturbation fields satisfy the following non-linear system:

$$\dot{\boldsymbol{\sigma}}^0 + \delta\dot{\boldsymbol{\sigma}} + b(\dot{p}^0 + \delta\dot{p})\mathbf{1} = \boldsymbol{E}^{\mathrm{d}} : \left[\dot{\boldsymbol{\epsilon}}^0 + \delta\dot{\boldsymbol{\epsilon}} - (\dot{\lambda}^0 + \delta\dot{\lambda})\frac{\partial F}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}^0 + \delta\boldsymbol{\sigma}, p^0 + \delta p, \boldsymbol{\chi}^0 + \delta\boldsymbol{\chi})\right] \tag{53}$$

$$\dot{p}^0 + \delta\dot{p} = M\left[\dot{\zeta}^0 + \delta\dot{\zeta} - (\dot{\lambda}^0 + \delta\dot{\lambda})\frac{\partial F}{\partial p}(\boldsymbol{\sigma}^0 + \delta\boldsymbol{\sigma}, p^0 + \delta p, \boldsymbol{\chi}^0 + \delta\boldsymbol{\chi}) - b\mathbf{1} : \left(\boldsymbol{E}^{\mathrm{d}}\right)^{-1} : \left(\dot{\boldsymbol{\sigma}}^0 + \delta\dot{\boldsymbol{\sigma}}\right) - \frac{b^2}{K^{\mathrm{d}}}(\dot{p}^0 + \delta\dot{p})\right] \tag{54}$$

$$\dot{\boldsymbol{\chi}}^0 + \delta\dot{\boldsymbol{\chi}} = -(\dot{\lambda}^0 + \delta\dot{\lambda})\boldsymbol{h} \cdot \frac{\partial F}{\partial \boldsymbol{\chi}}(\boldsymbol{\sigma}^0 + \delta\boldsymbol{\sigma}, p^0 + \delta p, \boldsymbol{\chi}^0 + \delta\boldsymbol{\chi}) \tag{55}$$

$$f(\boldsymbol{\sigma}^0 + \delta\boldsymbol{\sigma}, p^0 + \delta p, \boldsymbol{\chi}^0 + \delta\boldsymbol{\chi}) \leqslant 0, \quad \dot{\lambda}^0 + \delta\dot{\lambda} \geqslant 0, \quad f(\boldsymbol{\sigma}^0 + \delta\boldsymbol{\sigma}, p^0 + \delta p, \boldsymbol{\chi}^0 + \delta\boldsymbol{\chi})(\dot{\lambda}^0 + \delta\dot{\lambda}) = 0 \tag{56}$$

$$\operatorname{div}\delta\boldsymbol{\sigma} - (1-\phi)\rho_{\mathrm{s}}\delta\ddot{\boldsymbol{u}}_{\mathrm{s}} - \phi\rho_{\mathrm{f}}\delta\ddot{\boldsymbol{u}}_{\mathrm{f}} = \mathbf{0} \tag{57}$$

$$\operatorname{grad}\delta p + \rho_{\mathrm{f}}\delta\ddot{\boldsymbol{u}}_{\mathrm{f}} + \frac{\phi}{k}(\delta\dot{\boldsymbol{u}}_{\mathrm{f}} - \delta\dot{\boldsymbol{u}}_{\mathrm{s}}) = \mathbf{0} \tag{58}$$

$$\delta\boldsymbol{\epsilon}(\boldsymbol{u}) = \tfrac{1}{2}(\nabla\delta\boldsymbol{u}_{\mathrm{s}} + (\nabla\delta\boldsymbol{u}_{\mathrm{s}})^{\mathrm{T}}) \tag{59}$$

$$\delta\dot{\zeta} - k[\nabla^2\delta p + \operatorname{div}(\rho_{\mathrm{f}}\delta\ddot{\boldsymbol{u}}_{\mathrm{f}})] = 0 \tag{60}$$

Note that non-linearity comes only from the constitutive behaviour and that the partial differential part is linear.

Remark 4.1. The non-linear initial value problem (53)–(60) is not differentiable due to Kuhn–Tucker conditions. To cope with this difficulty, the reference solution whose stability is in question will be assumed in total loading during the whole loading process. Moreover, to simplify the analysis unloading from the perturbed solution is neglected. Note that the exclusive consideration of plastic loading regime, here as in the preceding section, precludes an assessment of the full consequences of the detected flutter instabilities since the oscillations of flutter solutions may lead to local plastic unloading. However the assumption of the so-called linear comparison solid is very common in literature and allows for a comparison with previous results.

4.2. Linearization

Because the perturbation is small, this non-linear problem is linearized around the reference solution (superscript 0). The linearization procedure and the derivation of the eigenvalue problem follow exactly the same lines as for the quasi-static case. Details may be found in Benallal and Comi (submitted for publication). The eigenvalue problem is obtained by seeking directly to the original system (53)–(60) solutions in the form

$$\delta \boldsymbol{X} = \tilde{\boldsymbol{X}} \exp(\mathrm{i}\xi \boldsymbol{n} \cdot \boldsymbol{x} + \eta(t - t_0)) \tag{61}$$

where $\boldsymbol{n}$ is a polarization direction, ξ the wave number of the perturbation mode and η may be related to the local rate of growth (in time) of the perturbation.

The eigenvalues η satisfy then the following algebraic system (from now on and for clarity, we omit here the subscript 0 related to the reference solution):

$$\left(\eta \boldsymbol{I} + \dot{\lambda}\boldsymbol{E}^{\mathrm{d}} : \frac{\partial^2 F}{\partial \boldsymbol{\sigma} \partial \boldsymbol{\sigma}}\right) : \tilde{\boldsymbol{\sigma}} + \left(\eta b \mathbf{1} + \dot{\lambda}\boldsymbol{E}^{\mathrm{d}} : \frac{\partial^2 F}{\partial \boldsymbol{\sigma} \partial p}\right)\tilde{p} + \dot{\lambda}\boldsymbol{E}^{\mathrm{d}} : \frac{\partial^2 F}{\partial \boldsymbol{\sigma} \partial \boldsymbol{\chi}} \cdot \tilde{\boldsymbol{\chi}} + \eta \boldsymbol{E}^{\mathrm{d}} : \frac{\partial F}{\partial \boldsymbol{\sigma}} \tilde{\dot{\lambda}} = \eta \boldsymbol{E}^{\mathrm{d}} : \tilde{\boldsymbol{\epsilon}} \tag{62}$$

$$\left(\eta b 1 : \boldsymbol{E}^{\mathrm{d}^{-1}} + \dot{\lambda}\frac{\partial^2 F}{\partial p \partial \boldsymbol{\sigma}}\right) : \tilde{\boldsymbol{\sigma}} + \left[\eta \frac{K^{\mathrm{u}}}{MK^{\mathrm{d}}} + \dot{\lambda}\frac{\partial^2 F}{\partial p \partial p}\right]\tilde{p} + \dot{\lambda}\frac{\partial^2 F}{\partial p \partial \boldsymbol{\chi}} \cdot \tilde{\boldsymbol{\chi}} + \eta \tilde{\dot{\lambda}}\frac{\partial F}{\partial p} = \eta \tilde{\zeta} \tag{63}$$

$$\dot{\lambda}\boldsymbol{h} \cdot \frac{\partial^2 F}{\partial \boldsymbol{\chi} \partial \boldsymbol{\sigma}} : \tilde{\boldsymbol{\sigma}} + \dot{\lambda}\boldsymbol{h} \cdot \frac{\partial^2 F}{\partial \boldsymbol{\chi} \partial p}\tilde{p} + \left(\eta \boldsymbol{I} + \dot{\lambda} \cdot \boldsymbol{h}\frac{\partial^2 F}{\partial \boldsymbol{\chi} \partial \boldsymbol{\chi}}\right) \cdot \tilde{\boldsymbol{\chi}} + \eta \boldsymbol{h} \cdot \frac{\partial F}{\partial \boldsymbol{\chi}} \tilde{\dot{\lambda}} = 0 \tag{64}$$

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} : \tilde{\boldsymbol{\sigma}} + \frac{\partial f}{\partial p}\tilde{p} + \frac{\partial f}{\partial \boldsymbol{\chi}} \cdot \tilde{\boldsymbol{\chi}} = 0 \tag{65}$$

$$\mathrm{i}\xi \tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{n} - (1 - \phi)\eta^2 \rho_{\mathrm{s}} \tilde{\boldsymbol{u}}_{\mathrm{s}} - \phi \eta^2 \rho_{\mathrm{f}} \tilde{\boldsymbol{u}}_{\mathrm{f}} = \boldsymbol{0} \tag{66}$$

$$\mathrm{i}\xi \tilde{p}\boldsymbol{n} + \eta^2 \rho_{\mathrm{f}} \tilde{\boldsymbol{u}}_{\mathrm{f}} + \frac{\phi}{k}\eta(\tilde{\boldsymbol{u}}_{\mathrm{f}} - \tilde{\boldsymbol{u}}_{\mathrm{s}}) = \boldsymbol{0} \tag{67}$$

$$\tilde{\boldsymbol{\epsilon}} = \frac{\mathrm{i}\xi}{2}(\tilde{\boldsymbol{u}}_{\mathrm{s}} \otimes \boldsymbol{n} + \boldsymbol{n} \otimes \tilde{\boldsymbol{u}}_{\mathrm{s}}) \tag{68}$$

$$\eta \tilde{\zeta} + k\xi^2 \tilde{p} - \mathrm{i}k\xi \eta^2 \rho_{\mathrm{f}} \tilde{\boldsymbol{u}}_{\mathrm{f}} \cdot \boldsymbol{n} = 0 \tag{69}$$

4.3. Eigenvalue problem

The eigenvalue problem in the general case is expressed by Eqs. (62)–(69). First, from (67), one can compute the fluid displacement as

$$\tilde{\boldsymbol{u}}_{\mathrm{f}} = \frac{\phi \eta \tilde{\boldsymbol{u}}_{\mathrm{s}} - \mathrm{i}\xi k \tilde{p}\boldsymbol{n}}{\eta(k\rho_{\mathrm{f}}\eta + \phi)} \tag{70}$$

Putting (70) in (69), one has $\tilde{\zeta}$ only in terms of the pressure amplitude $\tilde{p}$ and solid displacement $\tilde{\boldsymbol{u}}_{\mathrm{s}}$ as

$$\eta \tilde{\zeta} = -\eta \frac{\omega}{M}\tilde{p} + \mathrm{i}\xi \eta \phi \Omega \tilde{\mathbf{u}}_{\mathrm{s}} \cdot \boldsymbol{n} \tag{71}$$

where we have defined the non-dimensional parameters

$$\omega = \frac{Mk\xi^2}{\eta}\frac{\phi}{k\rho_f\eta + \phi} \tag{72}$$

$$\Omega = \frac{k\eta\rho_f}{k\rho_f\eta + \phi} \tag{73}$$

Now using (70) and (71), the balance of momentum (66) is written

$$\mathrm{i}\xi\tilde{\boldsymbol{\sigma}}\cdot\boldsymbol{n} + \mathrm{i}\xi\phi\Omega\tilde{p}\boldsymbol{n} - R\eta^2\tilde{\boldsymbol{u}}_s = \mathbf{0} \tag{74}$$

with

$$R = (1-\phi)\rho_s + \frac{\phi^2\rho_f}{k\rho_f\eta + \phi} \tag{75}$$

For later use, note that Ω and R coincide exactly with $\overline{\Omega}$ and $\overline{R}$, respectively, if one sets $\eta = -\mathrm{i}\xi c$.

The set of the four first equations of the eigenvalue problem (62)–(65) are put in the compact form (assuming $\eta \neq 0$ and using (71) to eliminate $\tilde{\zeta}$)

$$\mathscr{S}\bullet\tilde{\boldsymbol{Y}} = \left[\boldsymbol{S} + \frac{\dot{\lambda}}{\eta}\boldsymbol{T} + \frac{\omega}{M}\boldsymbol{P}\right]\bullet\tilde{\boldsymbol{Y}} = \tilde{\boldsymbol{E}} - \tilde{\dot{\lambda}}\frac{\partial F}{\partial \boldsymbol{Y}} \tag{76}$$

$$\left(\frac{\partial f}{\partial \boldsymbol{Y}}\right)^{\mathrm{T}}\bullet\tilde{\boldsymbol{Y}} = 0 \tag{77}$$

where we have introduced the following block matrices and vectors, whose entries are tensors:

$$\boldsymbol{T} = \begin{bmatrix} \frac{\partial^2 F}{\partial\boldsymbol{\sigma}\partial\boldsymbol{\sigma}} & \frac{\partial^2 F}{\partial\boldsymbol{\sigma}\partial p} & \frac{\partial^2 F}{\partial\boldsymbol{\sigma}\partial\boldsymbol{\chi}} \\ \frac{\partial^2 F}{\partial p\partial\boldsymbol{\sigma}} & \frac{\partial^2 F}{\partial p\partial p} & \frac{\partial^2 F}{\partial p\partial\boldsymbol{\chi}} \\ \frac{\partial^2 F}{\partial\boldsymbol{\chi}\partial\boldsymbol{\sigma}} & \frac{\partial^2 F}{\partial\boldsymbol{\chi}\partial p} & \frac{\partial^2 F}{\partial\boldsymbol{\chi}\partial\boldsymbol{\chi}} \end{bmatrix}, \quad \boldsymbol{S} = \begin{bmatrix} (\boldsymbol{E}^{\mathrm{d}})^{-1} & \frac{b}{3K^{\mathrm{d}}}\mathbf{1} & \mathbf{0} \\ \frac{b}{3K^{\mathrm{d}}}\mathbf{1} & \frac{K^{\mathrm{u}}}{MK^{\mathrm{d}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{h}^{-1} \end{bmatrix} \quad \text{and} \quad \boldsymbol{P} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{78}$$

$$Y = \begin{bmatrix} \tilde{\boldsymbol{\sigma}} \\ \tilde{p} \\ \tilde{\boldsymbol{\chi}} \end{bmatrix}, \quad \tilde{E} = \begin{bmatrix} \tilde{\boldsymbol{\epsilon}} \\ \mathrm{i}\xi\phi\Omega(\tilde{\boldsymbol{u}}_s\cdot\boldsymbol{n}) \\ \mathbf{0} \end{bmatrix}, \quad \frac{\partial F}{\partial \boldsymbol{Y}} = \begin{bmatrix} \frac{\partial F}{\partial\boldsymbol{\sigma}} \\ \frac{\partial F}{\partial p} \\ \frac{\partial F}{\partial\boldsymbol{\chi}} \end{bmatrix} \quad \text{and} \quad \frac{\partial f}{\partial \boldsymbol{Y}} = \begin{bmatrix} \frac{\partial f}{\partial\boldsymbol{\sigma}} \\ \frac{\partial f}{\partial p} \\ \frac{\partial f}{\partial\boldsymbol{\chi}} \end{bmatrix} \tag{79}$$

and the symbol $\bullet$ denotes a proper tensorial product ($\otimes$,: or $\cdot$) which gives meaning to the formal multiplications involved.

5. Unbounded growth of perturbations

We consider first in this section the critical conditions for unbounded growth of perturbations for the general set of constitutive equations given in Section 2. This corresponds to $\mathrm{Re}(\eta) \to \infty$. We also consider here the more general case where $|\eta| \to \infty$. This includes beside unbounded growth, unbounded decay ($\mathrm{Re}(\eta) \to -\infty$) and highly oscillating perturbations ($\mathrm{Re}(\eta)$ finite and $\mathrm{Im}(\eta) \to \infty$). In all these situations the problem simplifies greatly and we proceed directly from (62)–(69). In fact all the terms in $\dot{\lambda}/\eta$ can be dropped in the eigenvalue problem.

In this situation $\Omega \to 1$ and $R \to R_s$ already defined in (44). Eqs. (63) and (64) can be solved for $\tilde{\boldsymbol{\sigma}}$ and $\tilde{\boldsymbol{\chi}}$, respectively. Substitution of this result into the linearized yield condition (65) allows to compute the plastic multiplier. Finally, Eqs. (62) and (63) are used to obtain the amplitudes of the stress and mass content in terms of those of the solid strain and pore fluid pressure as

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{H}^{\mathrm{d}} : \tilde{\boldsymbol{\epsilon}} - \boldsymbol{K}\tilde{p} \tag{80}$$

$$\tilde{\zeta} = \boldsymbol{L} : \tilde{\boldsymbol{\epsilon}} + D\tilde{p} \tag{81}$$

where all the involved quantities have already been defined in Section 2. Note that the perturbation quantities satisfy exactly the rate constitutive equations since the coefficient tensors in (80) and (81) are exactly the same as those of (20) and (21). Substituting (80) and (81) into (66), (67) and (69) and using (68), one gets

$$-\xi^2(\boldsymbol{n}\cdot\boldsymbol{H}^{\mathrm{d}}\cdot\boldsymbol{n})\cdot\tilde{\boldsymbol{u}}_{\mathrm{s}} - \mathrm{i}\xi(\boldsymbol{K}\cdot\boldsymbol{n})\tilde{p} - (1-\phi)\eta^2\rho_{\mathrm{s}}\tilde{\boldsymbol{u}}_{\mathrm{s}} - \phi\eta^2\rho_{\mathrm{f}}\tilde{\boldsymbol{u}}_{\mathrm{f}} = \boldsymbol{0} \tag{82}$$

$$\mathrm{i}\xi\tilde{p}\boldsymbol{n} + \eta^2\rho_{\mathrm{f}}\tilde{\boldsymbol{u}}_{\mathrm{f}} + \frac{\phi}{k}\eta(\tilde{\boldsymbol{u}}_{\mathrm{f}} - \tilde{\boldsymbol{u}}_{\mathrm{s}}) = \boldsymbol{0} \tag{83}$$

$$\mathrm{i}\xi\eta(\boldsymbol{n}\cdot\boldsymbol{L})\cdot\tilde{\boldsymbol{u}}_{\mathrm{s}} + (\eta D + k\xi^2)\tilde{p} - \mathrm{i}k\xi\eta^2\rho_{\mathrm{f}}\tilde{\boldsymbol{u}}_{\mathrm{f}}\cdot\boldsymbol{n} = \boldsymbol{0} \tag{84}$$

Substitution of the fluid displacement (70) in (84) allows to get the pressure amplitude only in terms of the solid displacement $\tilde{\boldsymbol{u}}_{\mathrm{s}}$, i.e.

$$\tilde{p} = -\mathrm{i}\xi\eta\frac{(k\rho_{\mathrm{f}}\eta + \phi)(\boldsymbol{n}\cdot\boldsymbol{L})\cdot\tilde{\boldsymbol{u}}_{\mathrm{s}} - k\phi\rho_{\mathrm{f}}\eta\tilde{\boldsymbol{u}}_{\mathrm{s}}\cdot\boldsymbol{n}}{(k\rho_{\mathrm{f}}\eta + \phi)(\eta D + k\xi^2) - k^2\xi^2\rho_{\mathrm{f}}\eta} \tag{85}$$

Finally reporting (70) and (85) in (82), one gets the equation governing the solid displacement field $\tilde{\boldsymbol{u}}_{\mathrm{s}}$ as

$$\left[\xi^2\boldsymbol{n}\cdot\left(\boldsymbol{H}^{\mathrm{d}} + \frac{(\boldsymbol{K}-\phi\mathbf{1})\otimes(\boldsymbol{L}-\phi\mathbf{1})}{D+\frac{\omega}{M}}\right)\cdot\boldsymbol{n} + R_{\mathrm{s}}\eta^2\mathbf{1}\right]\tilde{\boldsymbol{u}}_{\mathrm{s}} = \boldsymbol{0} \tag{86}$$

From which the eigenvalue equation is obtained, i.e.

$$\det\left[\xi^2\boldsymbol{n}\cdot\left(\boldsymbol{H}^{\mathrm{d}} + \frac{(\boldsymbol{K}-\phi\mathbf{1})\otimes(\boldsymbol{L}-\phi\mathbf{1})}{D+\frac{\omega}{M}}\right)\cdot\boldsymbol{n} + R_{\mathrm{s}}\eta^2\mathbf{1}\right] = 0 \tag{87}$$

Note that, even though in this analysis we have considered the limit $|\eta| \to \infty$, the parameter ω is still present since its limit depends on the wavelength through ξ.

The solution in the form of harmonic waves (46) and the solution of the linearized perturbation problem (61) have the same form if we set

$$\eta = -\mathrm{i}\xi c \tag{88}$$

Since in this section we are concerned by unbounded growth, from (88) we have either $\xi \to \infty$ or $c \to \infty$. The second situation is shown to be excluded from condition (51) and one concludes that unbounded growth corresponds necessarily to the short wavelength limit $\xi \to \infty$. In this case, $\omega \approx -(M\phi/\rho_{\mathrm{f}}c^2)$ and Eq. (87) reduces exactly to the characteristic equation of acceleration waves (43). Therefore in order that unbounded growth ($\mathrm{Re}(\eta) \to \infty$) occurs, through (88), it is necessary and sufficient that zero or non-real acceleration wave speeds exist. In other words, unbounded growth of perturbations corresponds exactly to $c^2 \leqslant 0$ or c^2 complex. Thus the inception of unbounded growth is related to stationary discontinuities or to flutter instabilities. This result was given also by Simoes et al. (1999).

We denote by $h^{\mathrm{d}}_{\mathrm{sta}}$ the critical value of the drained hardening modulus corresponding to the emergence of stationary acceleration waves (or stationary discontinuities) and by $h^{\mathrm{d}}_{\mathrm{flu}}$ its critical value associated to the emergence of flutter (for acceleration waves). The critical value of the drained hardening modulus associated to unbounded growth of perturbation h^{d}_{∞} is therefore

$$h^{\rm d}_{\infty} = \max[h^{\rm d}_{\rm sta}, h^{\rm d}_{\rm flu}] \tag{89}$$

5.1. Inertia of the fluid neglected

As pointed out in Remark 2.2, for slow dynamic processes as those occurring in seismic analyses the inertial terms of the fluid can be neglected. This is obtained by setting $\rho_{\rm f} = 0$. Consequently $\Omega = 0$ and $R = R_{\rm s}$. One shows that the eigenvalue equation becomes here

$$\det\left[\xi^2 \boldsymbol{n}\cdot\left(\boldsymbol{H}^{\rm d} + \frac{\boldsymbol{K}\otimes\boldsymbol{L}}{D+\frac{k\xi^2}{\eta}}\right)\cdot\boldsymbol{n} + R_{\rm s}\eta^2\mathbf{1}\right] = 0 \tag{90}$$

5.2. Comparison with the quasi-static case

The quasi-static case is recovered by putting $\rho_{\rm s} = \rho_{\rm f} = 0$ in all the equations and hence $\Omega = R = 0$. The eigenvalue equation reads now

$$\det\left[\xi^2 \boldsymbol{n}\cdot\left(\boldsymbol{H}^{\rm d} + \frac{\boldsymbol{K}\otimes\boldsymbol{L}}{D+\frac{k\xi^2}{\eta}}\right)\cdot\boldsymbol{n}\right] = 0 \tag{91}$$

This is exactly the relation obtained directly in Benallal and Comi (submitted for publication) where the quasi-static case was considered in its own. As shown in that paper, this condition can be transformed into

$$\eta D \det[\boldsymbol{n}\cdot\boldsymbol{H}^{\rm u}\cdot\boldsymbol{n}] + k\xi^2 \det[\boldsymbol{n}\cdot\boldsymbol{H}^{\rm d}\cdot\boldsymbol{n}] = 0 \tag{92}$$

involving both the drained and the undrained acoustic tensors of the saturated porous medium.

Comparing the results here obtained in the dynamic context with those holding in the quasi-static case and coming from (92), two main differences emerge.

- While in the quasi-static context, as a consequence of (92), the eigenvalue η is always real (also when the skeleton behaviour is non-associative), the eigenvalue η can be complex in the dynamic case if the model is non-associative. Therefore while unbounded growth manifests itself only through divergence growth of perturbations in static, also flutter can appear in dynamic.
- In the dynamic context unbounded growth always corresponds to the short wavelength limit $\xi \to \infty$ (localized mode). On the contrary in the quasi-static case when unbounded growth corresponds to the singularity of the undrained acoustic tensor $\boldsymbol{n}\cdot\boldsymbol{H}^{\rm u}\cdot\boldsymbol{n} = 0$ all wavelength are possible (diffuse mode) as can be seen from (92).

6. Onset of growth of perturbation

6.1. Growth condition

We consider now the development and growth of perturbations and therefore all contributions in the linearized problem. Of particular interest is the onset of growth. This onset may occur in various ways. Indeed, when the behaviour of the skeleton is non-associative, the eigenvalues η may be complex. The transition from decaying to growing behaviour occurs either through divergence ($\eta = 0$) or through flutter instabilities ($\mathrm{Re}(\eta) = 0$). Here it is more convenient to use the eigenvalue problem in the form (74), (76) and (77).

The derivation is here much more complex and to proceed further, we consider in the following the simplifying assumption that the constitutive behaviour be such that some of the second order derivatives of the plastic potential be zero, namely:

$$\frac{\partial^2 F}{\partial \boldsymbol{\sigma} \partial p} = \frac{\partial^2 F}{\partial \boldsymbol{\sigma} \partial \boldsymbol{\chi}} = \frac{\partial^2 F}{\partial p \partial \boldsymbol{\chi}} = \frac{\partial^2 F}{\partial \boldsymbol{\sigma} \partial p} = \frac{\partial^2 F}{\partial \boldsymbol{\chi} \partial \boldsymbol{\chi}} = \mathbf{0} \tag{93}$$

Note that constitutive models fulfilling Eq. (93) are actually very common in the literature. An example will be analyzed in Section 7.

In this case, the eigenvalue problem simplifies and matrix $\mathscr{S}$, Eq. (76), takes the form

$$\mathscr{S} = \begin{bmatrix} \boldsymbol{G}^{-1} & \frac{b}{3K^{\mathrm{d}}}\mathbf{1} & \mathbf{0} \\ \frac{b}{3K^{\mathrm{d}}}\mathbf{1} & s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{h}^{-1} \end{bmatrix} \tag{94}$$

where

$$\boldsymbol{G} = \left[(\boldsymbol{E}^{\mathrm{d}})^{-1} + w \frac{\partial^2 F}{\partial \boldsymbol{\sigma} \partial \boldsymbol{\sigma}} \right]^{-1} \tag{95}$$

$$s = \frac{K^{\mathrm{d}} + b^2 M^{\omega}}{M^{\omega} K^{\mathrm{d}}} + w \frac{\partial^2 F}{\partial p^2}, \quad M^{\omega} = \frac{M}{1+\omega} \tag{96}$$

and $w = \dot{\lambda}/\eta$ is the non-dimensional parameter measuring the ratio between the rate of growth of the solution and the rate of growth of the perturbation. One can solve the system (76) for $\boldsymbol{Y}$

$$\boldsymbol{Y} = \mathscr{S}^{-1} \cdot \left(\tilde{\boldsymbol{E}} - \frac{\partial F}{\partial \boldsymbol{Y}} \tilde{\dot{\lambda}} \right) \tag{97}$$

where

$$\mathscr{S}^{-1} = \begin{bmatrix} \boldsymbol{E}^{\mathrm{g}} & -\frac{b}{3rK^{\mathrm{d}}}\boldsymbol{G} : \mathbf{1} & \mathbf{0} \\ -\frac{b}{3rK^{\mathrm{d}}}\boldsymbol{G} : \mathbf{1} & \frac{1}{r} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{h} \end{bmatrix} \tag{98}$$

$$\boldsymbol{E}^{\mathrm{g}} = \boldsymbol{G} + \frac{b^2}{9r(K^{\mathrm{d}})^2} \boldsymbol{G} : \mathbf{1} \otimes \mathbf{1} : \boldsymbol{G} \tag{99}$$

$$r = s - \frac{b^2 \mathbf{1} : \boldsymbol{G} : \mathbf{1}}{9(K^{\mathrm{d}})^2} \tag{100}$$

Note that, from definitions (99) and (100), one has:

$$\boldsymbol{E}^{\mathrm{g}} : \mathbf{1} = \frac{s}{r} \boldsymbol{G} : \mathbf{1} \tag{101}$$

Putting the value of $\boldsymbol{Y}$ (97) into the consistency condition (77) and using the above equality, one can compute the plastic multiplier $\tilde{\dot{\lambda}}$ as:

$$\tilde{\dot{\lambda}} = \frac{\left(\frac{\partial f}{\partial \boldsymbol{\sigma}} - \frac{b\frac{\partial f}{\partial p}}{3K^{\mathrm{d}}s}\mathbf{1}\right) : \boldsymbol{E}^{\mathrm{g}} : \tilde{\boldsymbol{\epsilon}} + \mathrm{i}\xi\phi\Omega\left(\frac{1}{r}\frac{\partial f}{\partial p} - \frac{b}{3K^{\mathrm{d}}r}\frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{G} : \mathbf{1}\right)(\tilde{\boldsymbol{u}}_{\mathrm{s}} \cdot \boldsymbol{n})}{\frac{\partial f}{\partial \boldsymbol{Y}}^{\mathrm{T}} \cdot [\mathscr{S}]^{-1} \cdot \frac{\partial F}{\partial \boldsymbol{Y}}} \tag{102}$$

The denominator, which will be denoted in the following by $H(w, \omega)$, can be transformed using the expression (98) and relation (101):

$$H(w,\omega) = \frac{\partial f}{\partial \boldsymbol{Y}}^{\mathrm{T}} \cdot [\mathscr{S}]^{-1} \cdot \frac{\partial F}{\partial \boldsymbol{Y}} = \left[\frac{\partial f}{\partial \boldsymbol{\sigma}} - \frac{b\frac{\partial f}{\partial p}}{3K^{\mathrm{d}}s}\mathbf{1}\right] : \boldsymbol{E}^{\mathrm{g}} : \left[\frac{\partial F}{\partial \boldsymbol{\sigma}} - \frac{b\frac{\partial F}{\partial p}}{3K^{\mathrm{d}}s}\mathbf{1}\right] + \frac{\frac{\partial f}{\partial p}\frac{\partial F}{\partial p}}{s} + \frac{\partial f}{\partial \boldsymbol{\chi}} \cdot \boldsymbol{h} \cdot \frac{\partial F}{\partial \boldsymbol{\chi}} \tag{103}$$

Introducing the drained hardening modulus h^{d} one has

$$H(w,\omega) = h^{\mathrm{d}} + \frac{\partial f}{\partial \boldsymbol{\sigma}} : G : \frac{\partial F}{\partial \boldsymbol{\sigma}} + \frac{1}{r}\left(\frac{\partial f}{\partial p} - \frac{b}{3K^{\mathrm{d}}}\frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{G} : \mathbf{1}\right)\left(\frac{\partial F}{\partial p} - \frac{b}{3K^{\mathrm{d}}}\frac{\partial F}{\partial \boldsymbol{\sigma}} : \boldsymbol{G} : \mathbf{1}\right) \tag{104}$$

The stress $\tilde{\boldsymbol{\sigma}}$ and pressure $\tilde{p}$ are easily obtained from (97) and (98), using (101):

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{E}^{\mathrm{g}}\left(\tilde{\boldsymbol{\epsilon}} - \tilde{\dot{\lambda}}\frac{\partial F}{\partial \boldsymbol{\sigma}}\right) - \frac{b}{3K^{\mathrm{d}}r}\left(\mathrm{i}\xi\phi\Omega(\tilde{\boldsymbol{u}}_{\mathrm{s}} \cdot \boldsymbol{n}) - \tilde{\dot{\lambda}}\frac{\partial F}{\partial p}\right)\boldsymbol{G} : \mathbf{1} \tag{105}$$

$$\tilde{p} = -\frac{b}{3rK^{\mathrm{d}}}1 : G : \left(\tilde{\boldsymbol{\epsilon}} - \tilde{\dot{\lambda}}\frac{\partial F}{\partial \boldsymbol{\sigma}}\right) + \frac{1}{r}\left(\mathrm{i}\xi\phi\Omega(\tilde{\boldsymbol{u}}_{\mathrm{s}} \cdot \boldsymbol{n}) - \tilde{\dot{\lambda}}\frac{\partial F}{\partial p}\right) \tag{106}$$

Finally reporting (105), (106) in the balance of momentum Eq. (74) and taking into account the expression of $\tilde{\dot{\lambda}}$ (102), we obtain the general condition governing the solid displacement $\tilde{\boldsymbol{u}}_{\mathrm{s}}$

$$[\boldsymbol{n} \cdot \boldsymbol{H}(\eta, \xi) \cdot \boldsymbol{n}] \cdot \tilde{\boldsymbol{u}}_{\mathrm{s}} = \mathbf{0} \tag{107}$$

where the modulus $\boldsymbol{H}(\eta, \xi)$ has the form

$$\boldsymbol{H}(\eta,\xi) = \xi^2\left[\boldsymbol{E} - \frac{\boldsymbol{P} \otimes \boldsymbol{Q}}{H}\right] + R\eta^2 \mathbf{1}\overline{\otimes}\mathbf{1} \tag{108}$$

In Eq. (108) the dependence on η and ξ comes from w and ω, the tensor product $\overline{\otimes}$ is such that $[\mathbf{1}\overline{\otimes}\mathbf{1}]_{ijhk} = \delta_{ik}\delta_{jh}$, we have set

$$\boldsymbol{E} = \boldsymbol{G} + \frac{1}{r}\left(\frac{b}{3K^{\mathrm{d}}}\boldsymbol{G} : \mathbf{1} - \phi\Omega\mathbf{1}\right) \otimes \left(\frac{b}{3K^{\mathrm{d}}}\boldsymbol{G} : \mathbf{1} - \phi\Omega\mathbf{1}\right) \tag{109}$$

$$\boldsymbol{P} = \boldsymbol{G} : \frac{\partial F}{\partial \boldsymbol{\sigma}} - \frac{1}{r}\left(\frac{\partial F}{\partial p} - \frac{b}{3K^{\mathrm{d}}}\frac{\partial F}{\partial \boldsymbol{\sigma}} : \boldsymbol{G} : \mathbf{1}\right)\left(\frac{b}{3K^{\mathrm{d}}}\boldsymbol{G} : \mathbf{1} - \phi\Omega\mathbf{1}\right) \tag{110}$$

$$\boldsymbol{Q} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{G} - \frac{1}{r}\left(\frac{\partial f}{\partial p} - \frac{b}{3K^{\mathrm{d}}}\frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{G} : \mathbf{1}\right)\left(\frac{b}{3K^{\mathrm{d}}}\boldsymbol{G} : \mathbf{1} - \phi\Omega\mathbf{1}\right) \tag{111}$$

and H was already defined in (103).

Eq. (107) gives the general condition for growth of perturbations. In the deformation path, growth of perturbations is then signaled by the existence of non-zero solutions $\tilde{\boldsymbol{u}}_{\mathrm{s}}$ with a wave number ξ (ω) associated to an eigenvalue η (w) with a positive real part. This happens when

$$\det[\boldsymbol{n} \cdot \boldsymbol{H}(\eta, \xi) \cdot \boldsymbol{n}] = 0 \tag{112}$$

The conditions in Section 5 are recovered in the limit $|\eta| \to \infty$ and therefore $w \to 0$. The terms in $\dot{\lambda}/\eta$ disappear and the acoustic tensor $\boldsymbol{n} \cdot \boldsymbol{H}(\eta, \xi) \cdot \boldsymbol{n}$ reduces to the one in Eq. (87).

We denote by $h^{\mathrm{d}}_{\mathrm{cr}}(\eta, \xi)$ the critical value of the drained hardening modulus associated to a rate of growth η and wave number ξ, i.e. the value associated to the first appearance in a loading process of a perturbed mode with wave number ξ and rate of growth η.

6.2. The long wavelength limit

The long wavelength corresponds to $\xi \to 0$. In this case, Eqs. (112) and (108) imply necessarily $\eta = 0$ unless the modulus H vanishes. For $\xi \to 0$, we have $\omega = 0$ for any finite η and therefore

$$r(w,0) = \frac{1}{M} + \frac{b^2}{K^{\mathrm{d}}}(1 - \mathbf{1} : \boldsymbol{G} : \mathbf{1}) + w\frac{\partial^2 F}{\partial p^2}.$$

From (104), one obtains directly the critical value of the hardening modulus as

$$h^{\mathrm{d}}_{\mathrm{cr}}(\eta, 0) = -\frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{G} : \frac{\partial F}{\partial \boldsymbol{\sigma}} - \frac{1}{r(w,0)}\left(\frac{\partial f}{\partial p} - \frac{b}{3K^{\mathrm{d}}}\frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{G} : \mathbf{1}\right)\left(\frac{\partial F}{\partial p} - \frac{b}{3K^{\mathrm{d}}}\frac{\partial F}{\partial \boldsymbol{\sigma}} : \boldsymbol{G} : \mathbf{1}\right) \tag{113}$$

Note that this limit depends on w through $\boldsymbol{G}$ and r. For $w = 0$, (113) is the undrained snap-back threshold, while for w infinite, it reduces to the drained snap-back threshold.

6.3. The short wavelength limit

When $\xi \to \infty$ and for any finite η (real or complex), both ω and $r \to \infty$. Therefore from Eqs. (108)–(111) we have

$$\boldsymbol{H}(\eta, \infty) = \boldsymbol{G} - \frac{\boldsymbol{G} : \frac{\partial F}{\partial \boldsymbol{\sigma}} \otimes \frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{G}}{h^{\mathrm{d}} + \frac{\partial F}{\partial \boldsymbol{\sigma}} : \boldsymbol{G} : \frac{\partial f}{\partial \boldsymbol{\sigma}}} \tag{114}$$

and therefore

$$h^{\mathrm{d}}_{\mathrm{cr}}(\eta, \infty) = \max_{\boldsymbol{n}}\left[\left(\boldsymbol{n} \cdot \boldsymbol{G} : \frac{\partial F}{\partial \boldsymbol{\sigma}}\right) \cdot (\boldsymbol{n} \cdot \boldsymbol{G} \cdot \boldsymbol{n})^{-1} \cdot \left(\frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{G} \cdot \boldsymbol{n}\right) - \frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{G} : \frac{\partial F}{\partial \boldsymbol{\sigma}}\right] \tag{115}$$

6.4. Divergence growth

Divergence growth corresponds to the transition $\eta = 0$. In this case $\Omega = 0$ and $w \to \infty$. From (112) and (108) we have either $\xi = 0$ or

$$\det\left[\boldsymbol{n} \cdot \left(\boldsymbol{E} - \frac{\boldsymbol{P} \otimes \boldsymbol{Q}}{H}\right) \cdot \boldsymbol{n}\right] = 0 \tag{116}$$

We assume that the limit $\boldsymbol{G}_\infty$ of $\boldsymbol{G}$ exists as $w \to \infty$. Two situations should be considered here:

- When $(\partial^2 F/\partial p^2) \neq 0$, r is unbounded and we obtain

$$\boldsymbol{H}(0, \xi) = \boldsymbol{G}_\infty - \frac{\boldsymbol{G}_\infty : \frac{\partial F}{\partial \boldsymbol{\sigma}} \otimes \frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{G}_\infty}{h^{\mathrm{d}} + \frac{\partial F}{\partial \boldsymbol{\sigma}} : \boldsymbol{G}_\infty : \frac{\partial f}{\partial \boldsymbol{\sigma}}} \tag{117}$$

  and one sees that this limit is independent of the wave number ξ and the associated critical hardening modulus is now

$$h^{\mathrm{d}}_{\mathrm{cr}}(0,\xi) = \max_{\boldsymbol{n}} \left[\left(\boldsymbol{n}\cdot\boldsymbol{G}_\infty : \frac{\partial F}{\partial\boldsymbol{\sigma}} \right) \cdot (\boldsymbol{n}\cdot\boldsymbol{G}_\infty\cdot\boldsymbol{n})^{-1} \cdot \left(\frac{\partial f}{\partial\boldsymbol{\sigma}} : \boldsymbol{G}_\infty\cdot\boldsymbol{n} \right) - \frac{\partial f}{\partial\boldsymbol{\sigma}} : \boldsymbol{G}_\infty : \frac{\partial F}{\partial\boldsymbol{\sigma}} \right] \tag{118}$$

- When

$$\frac{\partial^2 F}{\partial p^2} = 0, \quad r = \frac{1}{M^\omega} + \frac{b^2}{K^{\mathrm{d}}}\left(1 - \frac{\boldsymbol{1} : \boldsymbol{G}_\infty : \boldsymbol{1}}{9K^{\mathrm{d}}}\right)$$

and we have

$$\boldsymbol{H}(0,\xi) = \boldsymbol{G}_\infty + \frac{b^2}{9rK^{\mathrm{d}}}\boldsymbol{G}_\infty : \boldsymbol{1}\otimes\boldsymbol{G}_\infty : \boldsymbol{1} \\ - \frac{\left[\boldsymbol{G}_\infty : \frac{\partial F}{\partial\boldsymbol{\sigma}} - \frac{b}{3rK^{\mathrm{d}}}\left(\frac{\partial F}{\partial p} - \frac{b}{3K^{\mathrm{d}}}\frac{\partial F}{\partial\boldsymbol{\sigma}} : \boldsymbol{G}_\infty : \boldsymbol{1}\right)\boldsymbol{G}_\infty : \boldsymbol{1}\right] \otimes \left[\frac{\partial f}{\partial\boldsymbol{\sigma}} : \boldsymbol{G}_\infty - \frac{b}{3rK^{\mathrm{d}}}\left(\frac{\partial f}{\partial p} - \frac{b}{3K^{\mathrm{d}}}\frac{\partial f}{\partial\boldsymbol{\sigma}} : \boldsymbol{G}_\infty : \boldsymbol{1}\right)\boldsymbol{G}_\infty : \boldsymbol{1}\right]}{h^{\mathrm{d}} + \frac{\partial f}{\partial\boldsymbol{\sigma}} : \boldsymbol{G}_\infty : \frac{\partial F}{\partial\boldsymbol{\sigma}} + \frac{1}{r}\left(\frac{\partial f}{\partial p} - \frac{b}{3K^{\mathrm{d}}}\frac{\partial f}{\partial\boldsymbol{\sigma}} : \boldsymbol{G}_\infty : \boldsymbol{1}\right)\left(\frac{\partial F}{\partial p} - \frac{b}{3K^{\mathrm{d}}}\frac{\partial F}{\partial\boldsymbol{\sigma}} : \boldsymbol{G}_\infty : \boldsymbol{1}\right)} \tag{119}$$

Due to the presence of M^ω in the expression of r, the limit is now dependent on the wave number ξ. Expressions similar to (118) are found for the critical hardening modulus.

6.5. Transition through flutter growth

The analysis is too long to be reported here. Transition through flutter growth will be examined in the next section for the particular case of Drucker–Prager constitutive behaviour for the skeleton.

7. Application to Drucker–Prager elastic–plastic saturated porous solids

We apply in this section the results obtained in the former sections to constitutive equations (of Drucker–Prager type) corresponding to the following yield function f and plastic potential F:

$$f = \sqrt{J_2} + \mu(I_1 + 3\mu^* p) - \chi \tag{120}$$

$$F = \sqrt{J_2} + \beta(I_1 + 3\beta^* p) - \chi \tag{121}$$

$J_2 = \frac{1}{2}\boldsymbol{s} : \boldsymbol{s}$ is the second invariant of the stress deviator $\boldsymbol{s}$, $I_1 = \sigma_{kk}$ the hydrostatic stress, χ is the size of the yield surface, μ and β are the friction and dilatancy coefficients, μ^* and β^* are the coefficients of the plastic effective stresses in the loading function and in the plastic potential. Let us denote by s_j the principal values of the deviatoric stress tensor arranged so that $s_3 \leqslant s_2 \leqslant s_1$ and by $N_3, N_2 = N$ and N_1 their normalized values $N_j = s_j/(2J_2)^{1/2}$. One may check that N_1 and N_3 can be expressed in terms of N, with $-\frac{1}{\sqrt{6}} \leqslant N \leqslant \frac{1}{\sqrt{6}}$ as

$$N_1 = -\frac{N}{2} + \sqrt{\frac{1}{2} - \frac{3}{4}N^2}, \quad N_3 = -\frac{N}{2} - \sqrt{\frac{1}{2} - \frac{3}{4}N^2} \tag{122}$$

The associated drained and undrained tangent moduli are respectively

$$\boldsymbol{H}^{\mathrm{d}} = \boldsymbol{E}^{\mathrm{d}} - \frac{\left(\frac{G}{\sqrt{J_2}}\boldsymbol{s} + 3K^{\mathrm{d}}\beta\boldsymbol{1}\right) \otimes \left(\frac{G}{\sqrt{J_2}}\boldsymbol{s} + 3K^{\mathrm{d}}\mu\boldsymbol{1}\right)}{h^{\mathrm{d}} + G + 9K^{\mathrm{d}}\mu\beta} \tag{123}$$

$$\boldsymbol{H}^{\mathrm{u}} = \boldsymbol{E}^{\mathrm{u}} - \frac{\left(\frac{G}{\sqrt{J_2}}\boldsymbol{s} + 3K^{\mathrm{u}}\beta^{\mathrm{u}}\boldsymbol{1}\right) \otimes \left(\frac{G}{\sqrt{J_2}}\boldsymbol{s} + 3K^{\mathrm{u}}\mu^{\mathrm{u}}\boldsymbol{1}\right)}{h^{\mathrm{u}} + G + 9K^{\mathrm{u}}\mu^{\mathrm{u}}\beta^{\mathrm{u}}} \tag{124}$$

where we have defined the undrained friction and dilatancy factors μ^{u}, β^{u}

$$\beta^{\mathrm{u}} = \beta\left(1 - \frac{bM\beta^*}{K^{\mathrm{u}}}\right) \quad \mu^{\mathrm{u}} = \mu\left(1 - \frac{bM\mu^*}{K^{\mathrm{u}}}\right) \tag{125}$$

Assumptions (93) are fulfilled and we have

$$\frac{\partial^2 f}{\partial\boldsymbol{\sigma}\partial\boldsymbol{\sigma}} = \frac{\partial^2 F}{\partial\boldsymbol{\sigma}\partial\boldsymbol{\sigma}} = \frac{1}{2\sqrt{J_2}}\left(\boldsymbol{I} - \frac{\mathbf{1}\otimes\mathbf{1}}{3} - \frac{\boldsymbol{s}\otimes\boldsymbol{s}}{2J_2}\right) \tag{126}$$

Tensor $\boldsymbol{G}$ defined in Eq. (95), becomes for this model

$$\boldsymbol{G} = \left[\left(\boldsymbol{E}^{\mathrm{d}}\right)^{-1} + w\frac{\partial^2 F}{\partial\boldsymbol{\sigma}\partial\boldsymbol{\sigma}}\right]^{-1} = \left[\left(\boldsymbol{E}^{\mathrm{d}}\right)^{-1} + w\frac{1}{2\sqrt{J_2}}\left(\boldsymbol{I} - \frac{\mathbf{1}\otimes\mathbf{1}}{3} - \frac{\boldsymbol{s}\otimes\boldsymbol{s}}{2J_2}\right)\right]^{-1} \tag{127}$$

Using the Shermann–Morrison formula to invert the tensor in square brackets, after some algebraic manipulations and including for notation convenience the non-dimensional coefficient $G/(J_2)^{1/2}$ into w so that $w = \dot{\lambda}G/\eta(J_2)^{1/2}$, we get

$$\boldsymbol{G} = \frac{2G}{1+w}\boldsymbol{I} + \frac{wK^{\mathrm{d}} + (K^{\mathrm{d}} - \frac{2G}{3})}{1+w}\mathbf{1}\otimes\mathbf{1} + \frac{wG}{J_2(1+w)}\boldsymbol{s}\otimes\boldsymbol{s} \tag{128}$$

We observe in passing that $\boldsymbol{G}$ has limits when $w \to 0$ and when $w \to \infty$ and these are given respectively by

$$\boldsymbol{G}(0) = \boldsymbol{E}^{\mathrm{d}} \tag{129}$$

$$\boldsymbol{G}(\infty) = \boldsymbol{G}_\infty = K^{\mathrm{d}}\mathbf{1}\otimes\mathbf{1} + \frac{G}{J_2}\boldsymbol{s}\otimes\boldsymbol{s} \tag{130}$$

With $\mathbf{1} : \boldsymbol{G} = \boldsymbol{G} : \mathbf{1} = 3K^{\mathrm{d}}\mathbf{1}$ and, using (95) into (100) and (99), the following expressions for r and $\boldsymbol{E}$ are obtained

$$r = \frac{1}{M^\omega} = \frac{1+\omega}{M} \tag{131}$$

$$\boldsymbol{E} = \boldsymbol{G} + (b - \phi\Omega)^2 M^\omega \mathbf{1}\otimes\mathbf{1} \tag{132}$$

Noticing now that

$$\boldsymbol{G} : \frac{\partial f}{\partial\boldsymbol{\sigma}} = \boldsymbol{E}^{\mathrm{d}} : \frac{\partial f}{\partial\boldsymbol{\sigma}}, \quad \boldsymbol{G} : \frac{\partial F}{\partial\boldsymbol{\sigma}} = \boldsymbol{E}^{\mathrm{d}} : \frac{\partial F}{\partial\boldsymbol{\sigma}} \tag{133}$$

and using (132), the moduli $\boldsymbol{H}(\eta,\xi)$, Eq. (108), can be expressed explicitly as

$$\begin{aligned}\boldsymbol{H}(\eta,\xi) = \xi^2 \Bigg[&\frac{2G}{1+w}\boldsymbol{I} + \left((b-\phi\Omega)^2 M^\omega + \frac{wK^{\mathrm{d}} + (K^{\mathrm{d}} - \frac{2G}{3})}{1+w}\right)\mathbf{1}\otimes\mathbf{1} + \frac{wG}{J_2(1+w)}\boldsymbol{s}\otimes\boldsymbol{s} \\ &- \frac{\left(\frac{G}{\sqrt{J_2}}\boldsymbol{s} + 3K^\omega\beta^\omega\mathbf{1} + 3\beta(\beta^* - b)M^\omega\phi\Omega\mathbf{1}\right)\otimes\left(\frac{G}{\sqrt{J_2}}\boldsymbol{s} + 3K^\omega\mu^\omega\mathbf{1} + 3\mu(\mu^* - b)M^\omega\phi\Omega\mathbf{1}\right)}{h^\omega + G + 9\mu^\omega\beta^\omega K^\omega}\Bigg] \\ &+ R\eta^2\mathbf{1}\overline{\otimes}\mathbf{1}\end{aligned} \tag{134}$$

In (134) we have set

$$K^{\omega} = K^{\mathrm{d}} + b^2 M^{\omega} \tag{135}$$

$$\beta^{\omega} = \beta\left(1 - \frac{b\beta^* M^{\omega}}{K^{\omega}}\right) \tag{136}$$

$$\mu^{\omega} = \mu\left(1 - \frac{b\mu^* M^{\omega}}{K^{\omega}}\right) \tag{137}$$

$$h^{\omega} = h^{\mathrm{d}} + 9M^{\omega}\frac{K^{\mathrm{d}}\mu\mu^*\beta\beta^*}{K^{\omega}} \tag{138}$$

The acoustic tensor associated to (134) can be expressed as:

$$\boldsymbol{n}\cdot\boldsymbol{H}(\eta,\xi)\cdot\boldsymbol{n} = C_1\mathbf{1} + C_2\boldsymbol{n}\otimes\boldsymbol{n} + C_3(\boldsymbol{n}\cdot\boldsymbol{s})\otimes(\boldsymbol{s}\cdot\boldsymbol{n}) + C_4\boldsymbol{n}\otimes(\boldsymbol{s}\cdot\boldsymbol{n}) + C_5(\boldsymbol{n}\cdot\boldsymbol{s})\otimes\boldsymbol{n} \tag{139}$$

with

$$\begin{aligned}
C_1 &= \frac{G\xi^2}{1+w} + R\eta^2,\\
C_2 &= \xi^2\left[K^{\omega} + \frac{G}{3(1+w)} - \frac{9(K^{\omega}\beta^{\omega} + \beta(\beta^* - b)M^{\omega}\phi\Omega)(K^{\omega}\mu^{\omega} + \mu(\mu^* - b)M^{\omega}\phi\Omega)}{h^{\omega} + G + 9\mu^{\omega}\beta^{\omega}K^{\omega}}\right],\\
C_3 &= \xi^2\left[\frac{G}{J_2}\frac{w}{1+w} - \frac{G^2}{J_2 h^{\omega} + G + 9\mu^{\omega}\beta^{\omega}K^{\omega}}\right],\\
C_4 &= -3\xi^2\frac{K^{\omega}\beta^{\omega} + \beta(\beta^* - b)M^{\omega}\phi\Omega}{h^{\omega} + G + 9\mu^{\omega}\beta^{\omega}K^{\omega}}\frac{G}{\sqrt{J_2}},\\
C_5 &= -3\xi^2\frac{K^{\omega}\mu^{\omega} + \mu(\mu^* - b)M^{\omega}\phi\Omega}{h^{\omega} + G + 9\mu^{\omega}\beta^{\omega}K^{\omega}}\frac{G}{\sqrt{J_2}}.
\end{aligned} \tag{140}$$

Two eigenvectors of (139) can be looked for in the form

$$\boldsymbol{v} = A_1\boldsymbol{n} + A_2\boldsymbol{s}\cdot\boldsymbol{n} \tag{141}$$

leading to two eigenvalues of the acoustic tensor. Knowing the sum of the three eigenvalues by the trace of this acoustic tensor, one can get the third one. The determinant of the acoustic tensor is then obtained by the product of the three eigenvalues. This leads to

$$\begin{aligned}
\det(\boldsymbol{n}\cdot\boldsymbol{H}(\eta,\xi)\cdot\boldsymbol{n}) = C_1\Big\{&(C_1C_3 + C_2C_3 - C_4C_5)2J_2T + (C_4C_5 - C_2C_3)2J_2\Sigma^2\\
&+ C_1(C_4 + C_5)\sqrt{2J_2}\Sigma + C_1^2 + C_1C_2\Big\} = 0
\end{aligned} \tag{142}$$

where Σ is the normal component of the deviatoric stress and T is the amplitude of the normalized deviatoric stress vector

$$\Sigma = \frac{\boldsymbol{n}\cdot\boldsymbol{s}\cdot\boldsymbol{n}}{\sqrt{2J_2}} \quad T = \frac{(\boldsymbol{s}\cdot\boldsymbol{n})\cdot(\boldsymbol{s}\cdot\boldsymbol{n})}{2J_2} \tag{143}$$

Eq. (142) represents a parabola in the plane (T, Σ) (modified Mohr's plane) and can be solved in closed form for $h^{\mathrm{d}}_{\mathrm{cr}}(\eta,\xi)$ for real η, using the geometrical method (see Benallal and Comi, 1996). For complex η, the solution technique is much more complex. We limit the analysis here to unbounded growth and transition from decay to growth only.

7.1. Unbounded growth—stationary discontinuities

As shown in Section 5, this is given by the singularity of the drained acoustic tensor and therefore to localization under drained conditions. The associated critical conditions were given in Rudnicki and Rice (1975), Perrin and Leblond (1993), Benallal and Comi (1996). The results can be found in any of these references.

7.2. Unbounded growth—flutter regime

We have seen in Section 5 that unbounded growth was also possible at the onset of flutter (acceleration waves). The flutter phenomenon was investigated in great detail by Loret and Harireche (1991). The results of this analysis are thus not repeated here.

7.3. Onset of growth—divergence growth

For the constitutive equations adopted here, $(\partial^2 F/\partial p^2) = 0$ and the adequate limit is (119). With $r = 1/M^\omega$ and using the limit value $\boldsymbol{G}_\infty$ given by Eq. (130), we have

$$\boldsymbol{H}(0,\xi) = K^\omega \mathbf{1}\otimes\mathbf{1} + \frac{G}{J_2}\boldsymbol{s}\otimes\boldsymbol{s} - \frac{\left(\frac{G}{\sqrt{J_2}}\boldsymbol{s} + 3K^\omega\beta^\omega\mathbf{1}\right)\otimes\left(\frac{G}{\sqrt{J_2}}\boldsymbol{s} + 3K^\omega\mu^\omega\mathbf{1}\right)}{h^\omega + G + 9K^\omega\mu^\omega\beta^\omega} \tag{144}$$

We denote by h_0^{d} the critical value associated to $\boldsymbol{H}(0,\xi)$, i.e. the maximum values of h^{d} for which the eigenvalue η vanishes for the first time in a loading process. From Benallal and Comi (submitted for publication), we have

$$h_0^{\mathrm{d}} = \max_\omega\left[0, -9M^\omega\frac{K^{\mathrm{d}}\mu\mu^*\beta\beta^*}{K^\omega}\right] \tag{145}$$

7.4. Onset of growth—flutter regime

In the presence of non-associative behaviour for the skeleton, transition from decaying behaviour to growing behaviour of perturbations may also occur when two complex and conjugate eigenvalues cross the imaginary axis. We consider here this possibility in the short wavelength regime only. In this case, expression $\boldsymbol{H}(\eta,\infty)$, Eq. (114) is valid and the associated critical condition is (115). We suppose that the eigenvalue $\eta = \alpha + \mathrm{i}\kappa$ is complex, i.e. with κ real and non-zero. The solution of (115) in presence of complex eigenvalues is given in Benallal and Comi (submitted for publication) where a geometrical method was used. The optimization with respect to the normal $\boldsymbol{n}$ gives six different expressions for the hardening modulus h^{d}, each one valid within a defined range of material and loading parameters. Three solutions correspond to critical normals coincident with the principal directions of stress and the three others to a normal lying in each of the principal planes of stress. Our interest here is the transition from decay to growth of perturbations through $\mathrm{Re}(\eta) = 0$. This is shown to occur when the drained hardening modulus satisfies

$$h^{\mathrm{d}} = \max\left[h_1^{\mathrm{d}}, h_2^{\mathrm{d}}, h_3^{\mathrm{d}}, h_{12}^{\mathrm{d}}, h_{13}^{\mathrm{d}}, h_{23}^{\mathrm{d}}\right] \tag{146}$$

where for $i = 1,2,3$

$$\frac{h_i^{\mathrm{d}}}{G} = -\frac{(1+v)(18\mu\beta N_i^2 - 3\sqrt{2}N_i(\mu+\beta) + 1)}{3(1-2v)N_i^2 + (1+v)} \tag{147}$$

and for i, j, k distinct indices taking values 1, 2, 3,

$$\frac{h^{\mathrm{d}}_{ij}}{G} = \frac{(1+\nu)[8+12\beta\mu(\nu-5)+9\sqrt{2}N(\beta+\mu)(\nu-1)-4\nu+6N_k^2(\nu-2+6\beta\mu(1+\nu))+\sqrt{B}]}{6(4-2\nu+3N_k^2\ (-1+\nu^2))} \tag{148}$$

with

$$\begin{aligned} B = {} & 18\Big(10\beta\mu-3\mu^2-2\sqrt{2}N_k(\beta+\mu)(-1+12\beta\mu)-N_k^2(-2+24\beta\mu)-\beta^2\big(3+48\mu^2\big)\Big) \\ & +\big(4-2\nu+3N_k^2(-1+\nu^2)\big)+\Big(4(2+3\beta\mu(-5+\nu)-\nu)+9\sqrt{2}N_k(\beta+\mu)(-1+\nu) \\ & +6N_k^2(-2+\nu+6\beta\mu(1+\nu))\Big)^2 \end{aligned} \tag{149}$$

These expressions are valid only if they correspond to real critical normals $\boldsymbol{n}$ and if moreover κ is real and non-zero.

7.5. Illustration and discussion

The above results are illustrated in Fig. 1 for two different situations, namely for associative behaviour in Fig. 1a and for non-associative behaviour of the skeleton in Fig. 1b. The following sets of materials parameters, corresponding to a soil, have been chosen respectively for the two situations:

$$\mu = 0.08, \quad \mu^* = 1, \quad \beta = 0.08, \quad \beta^* = 1, \quad \nu = 0.3, \quad b = 1, \quad M = 7500\ \text{MPa}, \quad E = 60\ \text{MPa}$$

$$\mu = 0.08, \quad \mu^* = 1, \quad \beta = 0, \quad \beta^* = 1, \quad \nu = 0.3, \quad b = 1, \quad M = 7500\ \text{MPa}, \quad E = 60\ \text{MPa}.$$

The elastic parameters have been taken from Zienkiewicz and Shiomi (1984). The coefficients $b = \mu^* = \beta^* = 1$ correspond to Terzaghi's hypothesis, often used in soil mechanics. The parameter μ corresponds to an angle of internal friction of about 11°, reasonable for a clay soil. The dilatancy angle is equal to the angle of internal friction for the limit case of associative behaviour, while is assumed zero for the non-associative one.

In these figures, we plot critical conditions in terms of the normalized hardening modulus h^{d}/G versus the loading parameter $N_2 = N$. This parameter takes values between $-(1/\sqrt{6})$ and $1/\sqrt{6}$ for any type of loading. Note also that for a particular type of stress state, the loading process is represented by a vertical line directed downwards and along which we assume that the hardening modulus h^{d} decreases continuously.

In Fig. 1a corresponding to associative behaviour, transition from decay to growth of perturbation occurs only through divergence for $h^{\mathrm{d}} = 0$ (horizontal dot-dashed line) both under quasi-static and dynamic conditions. The onset of unbounded growth (continuous line) corresponds to the singularity of the drained acoustic tensor, i.e to localization under drained conditions (stationary discontinuities) and this happens at the same time again under quasi-static or dynamic conditions. We also report in Fig. 1a the onset of localization under undrained conditions (dashed line) which occurs after localization under drained conditions as shown in the general case in Benallal and Comi (2000).

The situation in Fig. 1b, corresponding to non-associative behaviour is totally different. First, transition from decay to growth occurs in various ways and three types of transition are identified:

- The first one is divergence growth and corresponds in Fig. 1b to segments AB and B′A′. One eigenvalue that was negative, crosses the imaginary axis and becomes positive. This occurs at $h^{\mathrm{d}} = 0$.

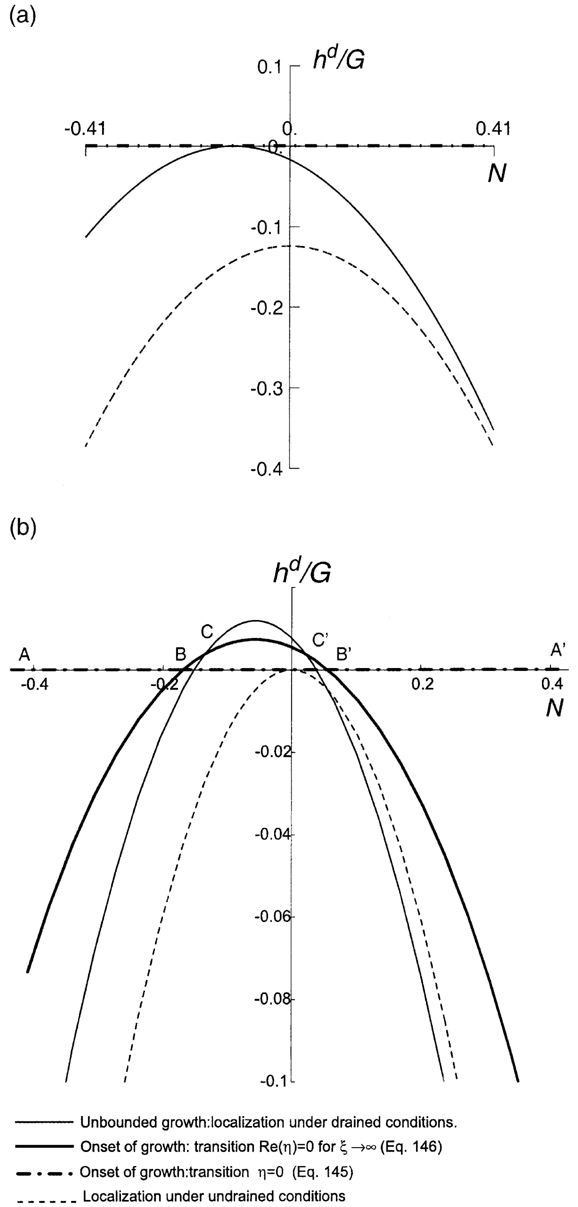

Fig. 1. Critical condition for the onset of growth and for unbounded growth of perturbations: (a) associative flow rule ($\beta = 0.08$); (b) non-associative flow rule ($\beta = 0.0$). Drucker–Prager skeleton $\mu = 0.08$, $\mu^* = 1$, $\beta^* = 1$, $\nu = 0.3$, $b = 1$, $M = 7500$ MPa, $E = 60$ MPa.

- The second corresponds to a flutter-type transition $\mathrm{Re}(\eta) = 0$ and corresponds in Fig. 1b to the parts BC and C′B′. There, two eigenvalues with negative real part cross the imaginary axis and their real part become positive.
- The last one corresponds to a singular transition in the sense that one eigenvalue that was infinitely negative (unbounded decay) becomes immediately infinitely positive (unbounded growth). This corresponds to the part CC′.

The onset of localization under undrained conditions is also reported as the dashed line. While it has no interest in the associative case, here it plays a role as it occurs before localization under drained conditions. Under quasi-static conditions, unbounded growth is linked to both types of localizations. Under dynamic conditions, the situation is slightly different and unbounded growth is linked either to stationary discontinuities or to the flutter phenomenon (acceleration waves).

8. Conclusions

Effects of inertia in the development and growth of instabilities were considered in this paper. Under quasi-static conditions, unbounded growth of perturbations is related to the conditions of localization under drained or undrained conditions, or in other words to the emergence of stationary discontinuities under drained or undrained conditions. Under dynamic conditions, unbounded growth is related either to the emergence of stationary discontinuities (and these are set only by drained properties) or to the appearance of the flutter phenomenon.

For associative behaviour localization under drained conditions always occurs before localization under undrained conditions (see Benallal and Comi, 2000). Moreover, as shown in Loret and Harireche (1991), flutter is excluded for associative flow. Therefore the onset of unbounded growth always corresponds to the singularity of the drained acoustic tensor either under quasi-static or dynamic conditions. The onset of growth of perturbations occurs here only through divergence growth.

In contrast, for non-associative flow rules, the undrained moduli may become critical before the drained one (see Benallal and Comi, 2000). Further, the flutter phenomenon may be possible (see Loret and Harireche, 1991). Unbounded growth is therefore no more related only to localization under drained conditions. Under quasi-static conditions, it may be related to localization under undrained conditions while under dynamic conditions, it is also related to flutter. Regarding the onset of growth, this may occur not only through divergence but also through flutter transition. One or the other depend on the details of the constitutive behaviour and the loading path.

Though the analysis was limited here to a homogeneous and infinite solid, a current study shows that many of the results contained in this paper remain valid when one considers heterogeneous situations and includes boundary conditions. These boundary conditions may bring however new features.

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